GENERALIZED COVER IDEALS AND THE PERSISTENCE PROPERTY

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Abstract. Let I be a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$, and consider the sets of associated primes $\operatorname{Ass}(I^s)$ for all integers $s \geq 1$. Many families of square-free monomial ideals are known to satisfy the persistence property, thus leading to the open question of whether or not all square-free monomial ideals satisfy this property. We introduce a family of square-free monomial ideals that can be associated to a finite simple graph G that generalizes the cover ideal construction. When G is a tree, we show our ideals satisfy the persistence property, thus providing more evidence for the open question. In fact, we can describe the elements of $\operatorname{Ass}(I^s)$ and explicitly determine the index of stability.

1. Introduction

Let I be an ideal of the polynomial ring $R = k[x_1, \ldots, x_n]$. A prime ideal $P \subseteq R$ is an *associated prime* of I if there exists an element $T \in R$ such that $I : \langle T \rangle = P$. The *set of associated primes* of I, denoted $\operatorname{Ass}(I)$, is the set of all prime ideals associated to I. We shall be interested in the sets $\operatorname{Ass}(I^s)$ as s varies. Brodmann [3] proved that there exists an integer s_0 such that $\operatorname{Ass}(I^s) = \operatorname{Ass}(I^{s_0})$ for all integers $s \geq s_0$. The least such integer s_0 is called the *index of stability*, and following [10], we denote it by $\operatorname{astab}(I)$. At least two problems follow from Brodmann's result: (1) determine $\operatorname{astab}(I)$ in terms of the invariants of R and I, and (2) does I satisfy the *persistence property*, that is, does $\operatorname{Ass}(I^s) \subseteq \operatorname{Ass}(I^{s+1})$ for all integers $s \geq 1$? Little is known about problem (1), although an upper bound on $\operatorname{astab}(I)$ when I is a monomial ideal is given by Hoa [12]. While not all ideals I satisfy the persistence property (see [3]), any normal ideal I, that is, I^s equals its integral closure $\overline{I^s}$ for all s, does satisfy this property (see [17]).

Interestingly, even when I is a square-free monomial ideal, we do not fully understand problems (1) and (2). However, the recent work of [1, 5, 10, 11, 14] has suggested possible answers. In particular, Herzog and Qureshi [10] posit that the bound $\operatorname{astab}(I) \leq \dim R - 1 = n - 1$ should hold for square-free monomial ideals I (this bound is significantly smaller than that given in [12]). A lower bound for $\operatorname{astab}(I)$ was given in [5] in terms of the chromatic number of a hypergraph constructed from the primary decomposition of I. On the other hand, a growing body of work appears to suggest that *all* square-free

Key words and phrases. monomial ideals, associated primes, trees, persistence property
2000 *Mathematics Subject Classification.* 13F20, 13A15, 05C25.
Version: Jan. 21, 2013 (submitted).

monomial ideals have the persistence property. This has been shown for edge ideals [14], the cover ideals of perfect graphs [5], and polymatroidal ideals [11]. Francisco, Hà, and the third author [6] have also described an approach to prove that all cover ideals of graphs have the persistence property that relies on proving a graph theoretic result. The survey of [16] may also be of interest. Note that it is known that non-square-free monomial ideals may not have the persistence property (see [9]).

In this paper, we introduce a family of square-free monomial ideals that can be associated to a finite simple graph G, and study the associated primes of their powers. More formally, suppose that G is a finite simple graph on vertex set $V_G = \{x_1, x_2, \ldots, x_n\}$ with edge set E_G. For any $x \in V_G$, we let $N(x) = \{y \mid \{x, y\} \in E_G\}$ denote the set of *neighbours of* x. By identifying the vertex x_i with the variable x_i in R, we define the following ideals.

Definition 1.1. Fix an integer $t \geq 1$. The *partial t-cover ideal* of G is the monomial ideal

$$J_t(G) = \bigcap_{x \in V_G} \left(\bigcap_{\{x_{i_1}, \ldots, x_{i_t}\} \subseteq N(x)} \langle x, x_{i_1}, \ldots, x_{i_t} \rangle \right).$$

When $t = 1$, our construction is simply the cover ideal of a finite simple graph G (see Section 2 for more details). Recall that a graph is a *tree* if it has no induced cycles. Our main result lends additional evidence to the proposed answers to (1) and (2).

Theorem 1.2. *Let $G = (V_G, E_G)$ be a tree on n vertices and fix any integer $t \geq 1$. Then the partial t-cover ideal $J_t(G)$ satisfies the persistence property. Furthermore*

$$\operatorname{astab}(J_t(G)) = \begin{cases} 1 & \text{if } t = 1 \\ \min\{s \mid s(t-1) \geq \Delta(G) - 1\} & \text{if } t > 1 \end{cases}$$

where $\Delta(G)$ is the maximal degree of G.

In fact, we prove a stronger result (Theorem 4.1) by determining the elements of $\operatorname{Ass}(J_t(G)^s)$ for all $s \geq 1$. Note that $\Delta(G) \leq n - 1$, so the upper bound suggested by Herzog and Qureshi also holds for this family.

Our paper is structured as follows. In Section 2, we review the required ingredients of associated primes and describe some of the properties of $J_t(G)$. In Section 3, we specialize to the case that $G = K_{1,n}$ is the star graph. These graphs will play an important role in our proof of Theorem 1.2; we also use these graphs to answer a question raised in [5]. Section 4 is devoted to the proof of our main result.

Acknowledgements. Some of the results of Section 3 first appeared in [2]. *Macaulay2* [8] was used for computer experiments. We thank T. Hà and C. Francisco for their feedback. The third author acknowledges the support of an NSERC Discovery Grant.

2. Preliminaries

We continue to use the terminology and definitions introduced in the previous section. Throughout this paper, $\mathcal{G}(I)$ denotes the unique set of minimal generators of a monomial ideal I. For any $W = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq V_G$, we let $x_W = x_{i_1} \cdots x_{i_s} \in R$.

We first explain the significance of the name partial t-cover ideal in Definition 1.1. A *vertex cover* of a graph G is a subset $W \subseteq V_G$ which satisfies the following property: for any $x \in V_G$, either $x \in W$ or $N(x) \subseteq W$. In other words, all the edges containing x are covered. We generalize this definition: a *partial t-cover* is a subset $W \subseteq V_G$ which satisfies the following property: for any $x \in V_G$, either $x \in W$ or there exists some subset $S \subseteq N(x)$ with $|S| = |N(x)| - t + 1$ and $S \subseteq W$. That is, for each $x \in V_G$, all, but perhaps $t-1$ of the edges containing x, are covered by W. When $t = 1$, this is simply the definition of a vertex cover. The following lemma justifies our choice of name for $J_t(G)$.

Lemma 2.1. *Let $G = (V_G, E_G)$ be a finite simple graph and $t \geq 1$ an integer. Then*

$$J_t(G) = \langle x_W \mid W \subseteq V_G \text{ is a partial } t\text{-cover}\rangle.$$

Proof. Let $m \in \mathcal{G}(J_t(G))$, and so $m = x_W$ for some $W \subseteq V_G$. Suppose W is not a partial t-cover. Then there exists a vertex x such that $x \notin W$, and for all $S \subseteq N(x)$ with $|S| = |N(x)|-t+1$, there is some $x_j \in S \setminus W$. We claim that there are t neighbours of x not in W. Let $S_1 = \{x_1, \ldots, x_{|N(x)|-t+1}\}$. Because W is not a partial t-cover, let $x_{i_1} \in S_1 \setminus W$. Set $S_2 = (S_1 \setminus \{x_{i_1}\}) \cup \{x_{|N(x)|-t+2}\}$. Again, W is not a partial t-cover, so there exists $x_{i_2} \in S_2 \setminus W$. We repeat t times and find t neighbours of x, say $\{x_{i_1}, \ldots, x_{i_t}\}$, that do not appear in W. It then follows that $m = x_W \notin \langle x, x_{i_1}, \ldots, x_{i_t}\rangle$ since none of these variables appear in x_W. But this contradicts the fact that $m \in J_t(G) \subseteq \langle x, x_{i_1}, \ldots, x_{i_t}\rangle$. Therefore W is a partial t-cover.

For the converse, let x_W be any square-free monomial which corresponds to a partial t-cover. Rewrite $J_t(G)$ as

$$J_t(G) = \left(\bigcap_{x \in W}\left(\bigcap_{\{x_{i_1},\ldots,x_{i_t}\}\subseteq N(x)} \langle x, x_{i_1}, \ldots, x_{i_t}\rangle\right)\right) \cap \left(\bigcap_{x \in V_G \setminus W}\left(\bigcap_{\{x_{i_1},\ldots,x_{i_t}\}\subseteq N(x)} \langle x, x_{i_1}, \ldots, x_{i_t}\rangle\right)\right).$$

If $x \in W$, then $x_W \in \langle x, x_{i_1}, \ldots, x_{i_t}\rangle$, so x_W is in the first intersection. If $x \notin W$, then there exists a subset $S \subseteq N(x)$ with $|N(x)| - t + 1$ elements such that $S \subseteq W$. But then for any subset $T \subseteq N(x)$ with $|T| = t$, $S \cap T \neq \emptyset$. This implies that $x_W \in \langle x, x_{i_1}, \ldots, x_{i_t}\rangle$ for each subset $\{x_{i_1}, \ldots, x_{i_t}\}$ of $N(x)$ of size t. So x_W is in the second intersection, thus completing the proof. □

Remark 2.2. The Alexander dual (see [15] for the definition) of $J_t(G)$ is also of interest:

$$I_t(G) := J_t(G)^\vee = \sum_{x \in V_G} \langle x x_{i_1} \cdots x_{i_t} \mid \{x_{i_1}, \ldots, x_{i_t}\} \subseteq N(x)\rangle.$$

If $t = 1$, then $I_1(G)$ is the edge ideal of G, and if $t = 2$, then $I_2(G)$ is the 2-path ideal of G (see [4] for the definition). The ideals $I_t(G)$ can be viewed as generalized edge ideals. In a future paper, we will investigate some of the properties of $I_t(G)$.

We turn to the relevant results on associated primes of square-free monomial ideals. Via the technique of localization, and using the fact that localization and taking powers commute, we simply need to determine when the maximal ideal is an associated prime of a monomial ideal. The following lemma justifies this reduction. The proof is similar to the proof of [5, Lemma 2.11], so is omitted. Given a graph $G = (V_G, E_G)$ and subset $P \subseteq V_G$, we write G_P for the *induced graph* on P, i.e., the graph with vertex set P, and edge set $E_{G_P} = \{e \in E_G \mid e \subseteq P\}$.

Lemma 2.3. *Let G be a graph on the vertex set $\{x_1, \ldots, x_n\}$, and let $J_t(G)$ be the partial t-cover ideal of G. The following are equivalent:*

(i) $P = \langle x_{i_1}, \ldots, x_{i_r}\rangle \in \mathrm{Ass}(J_t(G)^s)$ *in* $R = k[x_1, \ldots, x_n]$
(ii) $P = \langle x_{i_1}, \ldots, x_{i_r}\rangle \in \mathrm{Ass}(J_t(G_P)^s)$ *in* $R_P = k[x_{i_1}, \ldots, x_{i_r}]$.

The next lemma shows $P \in \mathrm{Ass}(J_t(G)^s)$ gives a necessary condition on the graph G_P.

Lemma 2.4. *Let G be a graph on the vertex set $\{x_1, \ldots, x_n\}$, and let $J_t(G)$ be the partial t-cover ideal of G. If $P = \langle x_{i_1}, \ldots, x_{i_r}\rangle \in \mathrm{Ass}(J_t(G)^s)$, then G_P is connected.*

Proof. By Lemma 2.3, it is enough to show that if $\langle x_1, \ldots, x_n\rangle \in \mathrm{Ass}(J_t(G)^s)$ for some s, then G is connected. Suppose G is not connected, i.e., $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \emptyset$. After relabeling the vertices, we can assume the vertices of G_1 are $\{y_1, \ldots, y_a\}$ and the vertices of G_2 are $\{z_1, \ldots, z_b\}$. If $m \in \mathcal{G}(J_t(G))$, then $m = m_y m_z$ where m_y is a square-free monomial in the y variables, and m_z is a square-free monomial in the z variables, and furthermore, we must have $m_y \in \mathcal{G}(J_t(G_1))$, and $m_z \in \mathcal{G}(J_t(G_2))$.

Because $\langle x_1, \ldots, x_n\rangle = \langle y_1, \ldots, y_a, z_1, \ldots, z_b\rangle$, and $\langle x_1, \ldots, x_n\rangle \in \mathrm{Ass}(J_t(G)^s)$, there exists a monomial $T \notin J_t(G)^s$ such that

$$\begin{aligned} Ty_1 &= m_1 \cdots m_s M \text{ with } m_i \in \mathcal{G}(J_t(G)) \\ &= m_{y,1}m_{z,1} \cdots m_{y,s}m_{z,s}M_yM_z \text{ with } m_i = m_{y,i}m_{z,i} \end{aligned}$$

where $m_{y,i} \in \mathcal{G}(J_t(G_1))$ and $m_{z,i} \in \mathcal{G}(J_t(G_2))$, and M_y (respectively M_z) is a monomial in the y variables (respectively the z variables). So, $T = (m_{z,1} \cdots m_{z,s}M_z)T'$ where T' is a monomial in the y variables. But we also know that $Tz_1 \in J_t(G)^s$, so a similar argument allows us to write $T = (u_{y,1} \cdots u_{y,s}U_y)T''$ where T'' is a monomial in the z variables, U_y is a monomial in the y variables, and each $u_{y,j} \in \mathcal{G}(J_t(G_1))$. But this means

$$T = (m_{z,1} \cdots m_{z,s}M_z)(u_{y,1} \cdots u_{y,s}U_y) = (u_{y,1}m_{z,1}) \cdots (u_{y,s}m_{z,s})U_yM_z.$$

Now each $u_{y,i}m_{z,i} \in \mathcal{G}(J_t(G))$, so $T \in J_t(G)^s$, a contradiction. Thus G is connected. □

Section 3 focuses on *star graphs* $G = K_{1,n}$. These are the graphs with vertex set $V_G = \{z, x_1, \ldots, x_n\}$ and edge set $E_G = \{\{z, x_i\} \mid 1 \leq i \leq n\}$. The generators of $J_t(K_{1,n})$, as described by the next lemma, follow directly from the definitions:

Lemma 2.5. *Let $G = K_{1,n}$ with $V = \{z, x_1, \ldots, x_n\}$, and let $n \geq t \geq 1$. Then*

$$J_t(G) = \langle z\rangle + \langle x_{j_1} \cdots x_{j_{n-t+1}} \mid \{j_1, \ldots, j_{n-t+1}\} \subseteq \{1, \ldots, n\}\rangle.$$

The next example explains what we know about $\mathrm{Ass}(J_t(K_{1,n})^s)$ when $t = 1$; the situation for $t \geq 2$ is explored in the next section.

Example 2.6. Let $G = K_{1,n}$ and $t = 1$. By Lemma 2.5, $J_1(G) = \langle z, x_1x_2 \cdots x_n \rangle$. But this is a complete intersection, so for all $s \geq 1$,

$$\mathrm{Ass}(J_1(G)^s) = \mathrm{Ass}(J_1(G)) = \{\langle z, x_i \rangle \mid 1 \leq i \leq n\}.$$

There are at least two ways to prove this result. For any complete intersection J, $J^s = J^{(s)}$, the s-th symbolic power of J (see [19]) and thus $\mathrm{Ass}(J^s) = \mathrm{Ass}(J)$ for all $s \geq 1$. Alternatively, Gitler, Reyes, and Villarreal have shown [7, Corollary 2.6] that $J_1(G)$ is normal, i.e., $J_1(G)^s = \overline{J_1(G)^s}$, whenever G is a bipartite graph, whence the conclusion again follows. Because $\mathrm{astab}(J_1(G)) = 1$, $J_1(G)$ has the persistence property.

3. Star graphs

Fix integers $n \geq t \geq 1$. In this section we will completely describe the sets $\mathrm{Ass}(J_t(G)^s)$ when $G = K_{1,n}$. We use our results to give a new answer to a question raised by Francisco, Hà, and the third author in [5]. Our main result is a corollary of the following theorem:

Theorem 3.1. *Fix integers $n \geq t \geq 1$ and let $G = K_{1,n}$ be the star graph on $V_G = \{z, x_1, \ldots, x_n\}$. Set $J_t = J_t(G)$. The following are equivalent:*

(i) $\langle z, x_1, \ldots, x_n \rangle \in \mathrm{Ass}(J_t^s)$
(ii) $s(t-1) \geq n-1$.

We postpone the proof, but record its consequences:

Corollary 3.2. *Fix integers $n \geq t \geq 1$ and let $G = K_{1,n}$ be the star graph on $V_G = \{z, x_1, \ldots, x_n\}$. For any $s \geq 1$,*

$$\mathrm{Ass}(J_t(G)^s) = \{\langle z, x_{i_1}, \ldots, x_{i_r} \rangle \mid t \leq r \leq \min\{n, s(t-1)+1\}\}.$$

Moreover,

$$\mathrm{astab}(J_t(G)) = \begin{cases} 1 & \text{if } t = 1 \\ \min\{s \mid s(t-1) \geq n-1\} & \text{if } t > 1. \end{cases}$$

Proof. The result on $\mathrm{astab}(J_t(G))$ follows from the first statement. Let $\mathcal{P}$ denote the set on the right hand side of the first statement. Let $P \in \mathrm{Ass}(J_t(G)^s)$. Because G_P is connected by Lemma 2.4, $P = \langle z, x_{i_1}, \ldots, x_{i_r} \rangle$, i.e., P cannot be generated by a subset of x variables. Note that this means that $G_P = K_{1,r}$ for some r. Either P is a minimal prime of $J_t(G)$, or contains a minimal prime of $J_t(G)$, thus showing showing that $t \leq r$. By Lemma 2.3, $\langle z, x_{i_1}, \ldots, x_{i_r} \rangle \in \mathrm{Ass}(J_t(G_P)^s)$, and so by Theorem 3.1, $s(t-1) \geq r-1$, i.e., $r \leq s(t-1)+1$. Also, it is clear that $r \leq n$, so $P \in \mathcal{P}$.

Conversely, suppose that $P = \langle z, x_{i_1}, \ldots, x_{i_r} \rangle \in \mathcal{P}$. Abusing notation, let $P \subseteq V_G$ denote the corresponding vertices. After localizing at P, $P \in \mathrm{Ass}(J_t(G_P)^s)$ by Theorem 3.1 since $s(t-1) \geq r-1$. Lemma 2.3 then gives $P \in \mathrm{Ass}(J_t(G)^s)$. $\square$

To prove Theorem 3.1 we require some information about our annihilator.

Lemma 3.3. *Fix integers $n \geq t \geq 1$ and let $G = K_{1,n}$ be the star graph on $V_G = \{z, x_1, \ldots, x_n\}$. Set $J_t = J_t(G)$. Suppose that there exists a monomial $T \in k[z, x_1, \ldots, x_n]$, $T \notin J_t^s$, such that $J_t^s : \langle T \rangle = \langle z, x_1, \ldots, x_n \rangle$. If $T = z^e T'$ where $z \nmid T'$, then $T \mid z^e(x_1 \cdots x_n)^{s-e-1}$.*

Proof. It suffices to prove that $T' | (x_1 \cdots x_n)^{s-e-1}$. Suppose that there exists some x_i such that $x_i^{s-e} | T'$. Now $x_i T = z^e x_i T' \in J_t^s$, so

$$z^e x_i T' = m_1 m_2 \cdots m_s M \text{ with } M \in k[z, x_1, \ldots, x_n] \text{ and } m_i \in \mathcal{G}(J_t).$$

We cannot have $x_i | M$. If it did, then we could cancel x_i from both sides and have $T = z^e T' = m_1 \cdots m_s (M/x_i) \in J_t^s$, which contradicts the fact that $T \notin J_t^s$. So, the variable x_i appears at least $s - e + 1$ times in $z^e x_i T'$, and thus, must appear in at least $s - e + 1$ of $m_1, \ldots, m_s$, because each m_j is square-free. In particular, we can assume $m_1 = x_i m_1'$. This means at most $e - 1$ of $m_1, \ldots, m_s$ can be equal to z (no minimal generator of J_t is divisible by both z and x_i by Lemma 2.5). So, z must divide M, i.e., $M = zM'$. So, to summarize,

$$z^e x_i T' = m_1 m_2 \cdots m_s M = (x_i m_1') m_2 \cdots m_s (zM').$$

If we cancel x_i from both sides, we get

$$T = z^e T' = (m_1') m_2 \cdots m_s (zM').$$

But $m_2, \ldots, m_s, z \in \mathcal{G}(J_t)$, which means $T \in J_t^s$. This is our desired contradiction. $\square$

We are now ready to prove Theorem 3.1.

Proof. (of Theorem 3.1) Note that if $t = 1$, then Example 2.6 implies $\langle z, x_1, \ldots, x_n \rangle \in \operatorname{Ass}(J_1(G)^s)$ if and only if $n = 1$ if and only if $0 = s(t-1) \geq n - 1$. So, we assume $t > 1$.

$(i) \Rightarrow (ii)$. If $\langle z, x_1, \ldots, x_n \rangle \in \operatorname{Ass}(J_t^s)$, then there exists a monomial $T \notin J_t^s$ such that $J_t^s : \langle T \rangle = \langle z, x_1, \ldots, x_n \rangle$. Rewrite T as $T = z^e T'$ where $z \nmid T'$. We now claim that

$$z^e (x_1 \cdots x_{n-t+2})^{s-e} (x_{n-t+3} \cdots x_n)^{s-e-1} \in J_t^{s+1}. \tag{3.1}$$

Indeed, by Lemma 3.3, $z^e T' | z^e (x_1 \cdots x_n)^{s-e-1}$. Now $x_1 z^e T' \in J_t^s$, which means

$$z^e x_1^{s-e} (x_2 \cdots x_n)^{s-e-1} \in J_t^s.$$

But $x_2 \cdots x_{n-t+2} \in J_t$, so multiplying these two elements together gives us the desired element in J_t^{s+1}.

We proceed by a degree argument. By (3.1) there exist generators $m_1, \ldots, m_{s+1}$ of J_t such that

$$z^e (x_1 \cdots x_{n-t+2})^{s-e} (x_{n-t+3} \cdots x_n)^{s-e-1} = m_1 \cdots m_{s+1} M.$$

By Lemma 2.5, f of these generators are of the form z, and the remaining $s + 1 - f$ generators are of degree $n - t + 1$ and have the form $x_{j_1} \cdots x_{j_{n-t+1}}$ for some $\{j_1, \ldots, j_{n-t+1}\} \subseteq$

$\{1, \ldots, n\}$. Note that we must have $f \leq e$, and thus, looking at the degree of the generators in the x variables, we must have

$$(s+1-f)(n-t+1) \leq (n-t+2)(s-e)+(t-2)(s-e-1) = (s-e)n-(t-2).$$

Expanding out the left hand side gives

$$sn - st + s + n - t + 1 - fn + ft - f \leq sn - en - t + 2.$$

Removing sn and $-t$ from both sides and using the fact that $-en \leq -fn$ and $0 \leq f(t-1)$ gives $-st+s+n \leq 1$, which implies $s(t-1) \geq n-1$, as desired.

$(ii) \Rightarrow (i)$ Let $s_0 = \min\{s \mid s(t-1) \geq n-1\}$. We first show that $\langle z, x_1, \ldots, x_n\rangle \in \mathrm{Ass}(J_t^{s_0})$.

We construct our annihilator as follows. Write out the variables $x_1, \ldots, x_n$ as a repeating sequence, i.e.,

$$x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, x_1, \ldots. \tag{3.2}$$

Let T be the product of the first $s_0(n-t+1)-1$ variables in this sequence, that is,

$$T = \underbrace{x_1x_2\cdots x_nx_1x_2\cdots x_nx_1\cdots x_j}_{s_0(n-t+1)-1}.$$

The monomial $T \notin J_t^{s_0}$. We can see this by a degree argument because J_t is generated by monomials in the x variables of degree $n-t+1$.

We make the crucial observation that the index j of the last variable in T has the property that $n-t+1 \leq j \leq n$. To see this, note that after $n-t+1$ steps in the sequence (3.2) we are at vertex x_{n-t+1}, after $2(n-t+1)$ steps in the sequence (3.2), we are at the vertex $x_{n-2(t-1)} = x_{n-2t+2}$, after $3(n-t+1)$ steps, we are at x_{n-3t+3}, ..., and finally, after $(s_0-1)(n-t+1)$ steps, we are at vertex $x_{n-(s_0-1)(t-1)} = x_{n-s_0t+s_0+t-1}$. By our choice of s_0, $-s_0t+s_0 \leq -n+1$, so $n-s_0t+s_0+t-1 \leq t$. In fact, after (s_0-1) steps of size $(n-t+1)$ in our sequence (3.2), this is the first time we arrive at an index $\leq t$. At the same time, by our choice of s_0, we have $(s_0-1)(t-1) < n-1$, so we are at an index ≥ 1. When constructing T, we go an additional $n-t$ steps in the sequence. This means that we arrive at an index between $n-t+1$ and n.

We next show $J_t^{s_0} : \langle T\rangle = \langle z, x_1, \ldots, x_n\rangle$. Now $zT \in J_t^{s_0}$. To see this, note that z is a minimal generator of J_t, and every $n-t+1$ consecutive variables in (3.2) is also a generator of J_t. Thus, the product of the first $(s_0-1)(n-t+1)$ elements of (3.2) is in $J_t^{s_0-1}$, and so $z \in J_t^{s_0} : \langle T\rangle$.

Now take x_i with $i \in \{1, \ldots, n\}$. To show $x_iT \in J_t^{s_0}$, take the first $s_0(n-t+1)-1$ variables in (3.2), and insert x_i after its first appearance, i.e.,

$$x_1, x_2, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, x_1, \ldots, x_j.$$

Think of these variables as being placed around a circle. Starting at the second x_i, move around the circle, grouping $n-t+1$ variables together. Because we have $s_0(n-t+1)$ variables, we end up with s_0 groups. Because the index of j is between $n-t+1$ and n,

each group will consist of $n-t+1$ distinct variables, and thus, by Lemma 2.5, when we multiply each group of $n-t+1$ distinct variables together, we have a generator of J_t. But this means that $x_iT \in J_t^{s_0}$ since x_iT is expressed as a product of s_0 generators. Thus, $\langle z, x_0, \ldots, x_n\rangle \subseteq J_t^{s_0} : \langle T\rangle \subsetneq \langle 1\rangle$, which completes the proof for the case s_0.

Now suppose that $s > s_0$. Let $e = s - s_0$ and let T be as above. We will show that $J_t^s : \langle z^eT\rangle = \langle z, x_1, \ldots, x_n\rangle$. By a degree argument $z^eT \notin J_t^s$, but $z(z^eT) \in J_t^s$ because, as noted above, $T \in J_t^{s_0-1}$ and $z^{e+1} \in J_t^{e+1}$. Similarly, $x_iz^eT \in J_t^s$ because $z^e \in J_t^e$, and as above, $x_iT \in J_t^{s_0}$. Hence $J_t^s : \langle z^eT\rangle = \langle z, x_1, \ldots, x_n\rangle$. □

3.1. **An application.** Corollary 3.2 allows us to answer a question raised by Francisco, Hà, and the third author [5]. We first recall some terminology.

A *hypergraph* $\mathcal{H}$ is a pair of sets $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ where $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{E}$ is a collection of subsets $\{E_1, \ldots, E_t\}$ with each $E_i \subseteq \mathcal{X}$. We call $\mathcal{H}$ a *simple* hypergraph if $|E_i| \geq 2$ for all i, and if $E_i \subseteq E_j$, then $i = j$. (When each $|E_i| = 2$, then $\mathcal{H}$ is a finite simple graph.) As in the case of graphs, we say a subset $W \subseteq \mathcal{X}$ is a *vertex cover* if $W \cap E \neq \emptyset$ for all $E \in \mathcal{E}$. In a manner analogous to the cover ideal, we can define the cover ideal of $\mathcal{H}$:

$$J(\mathcal{H}) = \langle x_W \mid W = \{x_{i_1}, \ldots, x_{i_t}\} \subseteq \mathcal{X} \text{ is a vertex cover}\rangle.$$

A *colouring* of $\mathcal{H}$ is an assignment of a colour to each vertex of $\mathcal{X}$ so that no edge E is mono-coloured, i.e., each edge must contain at least two vertices of different colours. The *chromatic number* of $\mathcal{H}$, denoted $\chi(\mathcal{H})$, is the least number of colours required to colour $\mathcal{H}$. The chromatic number provides a lower bound on the index of stability of $J(\mathcal{H})$.

Theorem 3.4 ([5, Corollary 4.9]). *For any finite simple hypergraph $\mathcal{H}$,*

$$\chi(\mathcal{H}) - 1 \leq \operatorname{astab}(J(\mathcal{H})).$$

It was asked in [5, Question 4.10] if for each $m \geq 1$, there exists a hypergraph $\mathcal{H}_m$ with $\chi(\mathcal{H}_m) - 1 + m \leq \operatorname{astab}(J(\mathcal{H}_m))$, that is, could the index of stability be arbitrarily larger than the chromatic number. Wolff [18] showed that this is the case, even if $\mathcal{H}$ is a finite simple graph. Wolff's family of graphs requires $5m-1$ vertices. We can use Corollary 3.2 to give another answer to this question which only requires $m+3$ vertices.

Theorem 3.5. *Fix an $m \geq 1$, and let $\mathcal{H}_m = (\mathcal{X}_m, \mathcal{E}_m)$ where $\mathcal{X}_m = \{z, x_1, \ldots, x_{m+2}\}$ and $\mathcal{E}_m = \{\{z, x_i, x_j\} \mid 1 \leq i < j \leq m+2\}$. Then*

$$\chi(\mathcal{H}_m) - 1 + m \leq \operatorname{astab}(J(\mathcal{H}_m)).$$

Proof. First, $\chi(\mathcal{H}_m) = 2$ because each x_i can be assigned the same colour, and z can be given a different colour. Note that $J(\mathcal{H}_m) = J_2(K_{1,m+2})$. By Corollary 3.2, $\operatorname{astab}(J(\mathcal{H}_m)) = \operatorname{astab}(J_2(K_{1,m+2})) \geq m+1 = (2-1)+m = \chi(\mathcal{H}_m) - 1 + m$. □

4. Associated primes of Generalized cover ideals of trees

In this section we completely determine the associated primes of the ideals $J_t(\Gamma)^s$ when Γ is a *tree*, that is, a graph with no induced cycles. Theorem 1.2 will follow directly from this result. We begin by stating the main theorem of this section:

Theorem 4.1. *Fix an integer $t \geq 1$ and let Γ be a tree on n vertices. Then for all $s \geq 1$,*

$$\operatorname{Ass}(J_t(\Gamma)^s) = \{P = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_r} \rangle \mid \Gamma_P = K_{1,r} \text{ with } t \leq r \leq \min\{n, s(t-1)+1\}\}.$$

In other words, a prime is associated to $J_t(\Gamma)^s$ if and only if the corresponding induced subgraph in Γ is a star of a particular size.

We require the following lemma which can be found in [13, Proposition 4.1]). This lemma will gives us some insight into the generators of $J_t(\Gamma)$.

Lemma 4.2. *For any tree Γ, there exists a vertex x such that all, but possibly one, of its neighbours have degree* 1*.*

We fix some notation to be used throughout the remainder of this paper. Let Γ be a tree, and let x be the vertex of Lemma 4.2 with neighbours $y_1, \ldots, y_d$. We can assume that $\deg y_1 = \cdots = \deg y_{d-1} = 1$ and $\deg y_d \geq 1$. Using this notation, we have:

Lemma 4.3. *Let Γ be a tree with partial t-cover ideal $J_t(\Gamma)$. If $m \in \mathcal{G}(J_t(\Gamma))$, then m has one of the following forms:*

(i) $m = y_{i_1} \cdots y_{i_{d-t+1}} m'$
(ii) $m = xm'$
(iii) $m = xy_d m'$

where in each case, m' is not divisible by any of the variables $y_1, \ldots, y_d$, x.

Proof. By Lemma 2.1, the minimal generators of $J_t(\Gamma)$ correspond to the minimal partial t-covers of Γ. The result will follow if we look at the corresponding statement for minimal partial t-covers of Γ.

Let W be a minimal partial t-cover of Γ. First, suppose that $x \notin W$. By definition, W must contain a subset $S \subseteq N(x)$ of size $|N(x)| - t + 1 = d - t + 1$. Because $N(x) = \{y_1, \ldots, y_d\}$, let us say that $S = \{y_{i_1}, \ldots, y_{i_{d-t+1}}\}$. It now suffices to show that $W \setminus S$ does not contain any other neighbours of x. If $t = 1$, then $S = N(x)$, so this is clear. So, suppose that $t \geq 2$, and suppose that there is some $y_j \in N(x) \cap (W \setminus S)$. There are two cases to consider: $j \neq d$ and $j = d$.

If $j \neq d$, then Lemma 4.2 gives $\deg y_j = 1$. We claim that $(W \setminus \{y_j\})$ is also a partial t-cover of Γ, thus contradicting the minimality of W. Indeed, take any vertex z of Γ. Because y_j is only adjacent to x, for any vertex $z \notin \{y_j, x\}$, either z is in $(W \setminus \{y_j\}) \subseteq W$ or all but perhaps $t-1$ of the neighbours of z are in $(W \setminus \{y_j\}) \subseteq W$. We know that $x \notin W$, but because $S \subseteq (W \setminus \{y_j\}) \subseteq W$, we know that all but perhaps $t-1$ of the neighbours of x are in $(W \setminus \{y_j\})$. Finally, although $y_j \notin W$, all but perhaps $t - 1 \geq 2 - 1 = 1$ of

its neighbours belong to W. But since y_j only has the neighbour x, $(W \setminus \{y_j\})$ is also a partial t-cover. If $j = d$, then we can simply repeat the above argument to show that $(W \setminus \{y_{i_1}\})$ (remove one the vertices of S, but keep y_d) creates a smaller partial t-cover.

Now consider the case that $x \in W$. It suffices to show that $\{y_1, \ldots, y_{d-1}\} \cap W = \emptyset$. Then we will have the form (ii) if $y_d \notin W$, and the form (iii) if $y_d \in W$. Suppose that $y_j \in \{y_1, \ldots, y_{d-1}\} \cap W$. We claim that $(W \setminus \{y_j\})$ would also be a partial t-cover. By Lemma 4.2, $\deg y_j = 1$, and y_j is only adjacent to x. As argued above, for any vertex $z \notin \{y_j, x\}$, either z or all but perhaps $t-1$ of its neighbours will belong to $(W \setminus \{y_j\})$. The vertex x is in $(W \setminus \{y_j\})$, and as for y_j, although $y_j \notin (W \setminus \{y_j\})$, the unique edge containing y_j is covered by x. So $(W \setminus \{y_j\})$ is a partial t-cover, contradicting the minimality of W. □

Proof. (of Theorem 4.1) Let $\mathcal{P}$ denote the set on the right. Lemma 2.3 and Corollary 3.2 imply that every induced star graph of Γ of the appropriate size will contribute an associated prime; more precisely, we already have $\mathcal{P} \subseteq \operatorname{Ass}(J_t(\Gamma)^s)$. It therefore suffices to show that if $P \in \operatorname{Ass}(J_t(\Gamma)^s)$, then Γ_P is a star graph. Corollary 3.2 and Lemma 2.3 then imply the condition on the size of the star graph, thus showing $P \in \mathcal{P}$.

We let $J = J_t(\Gamma)$. If $P \in \operatorname{Ass}(J^s)$, by Lemma 2.3 we can assume that $\Gamma_P = \Gamma$ and by Lemma 2.4, we can assume that Γ is connected. Because Γ is a tree, so is Γ_P. So, we can apply Lemma 4.2. That is, we can assume that there is a vertex x with neighbours $y_1, \ldots, y_d$ such that $\deg y_1 = \cdots = \deg y_{d-1} = 1$, and $\deg y_d \geq 1$ in Γ_P. It suffices to show that $\deg y_d = 1$. Since Γ_P is connected, this would mean $\Gamma_P = K_{1,d}$.

So, suppose y_d has a neighbour, say $w \neq x$. We thus have $P = \langle y_1, \ldots, y_d, x, w, \ldots \rangle$. We now want to build a contradiction from this information.

Since $P \in \operatorname{Ass}(J^s)$, there exists a monomial $T \notin J^s$ such that $J^s : \langle T \rangle = P$. Because $w \in P$,

$$Tw = m_1 \cdots m_s M \text{ with } m_i \in \mathcal{G}(J).$$

By Lemma 4.3, a generator of J has one of three forms. Let's say that a of $m_1, \ldots, m_s$ are of type (i), b of $m_1 \ldots, m_s$ are of type (ii), and c are of type (iii). We then have

$$T = T' y_1^{e_1} y_2^{e_2} \cdots y_{d-1}^{e_{d-1}} y_d^{e_d + c} x^{b+c}$$

where $e_1 + \cdots + e_d = (d - t + 1)a$ and $a + b + c = s$. Without loss of generality we may assume that $e_1 = \max\{e_1, \ldots, e_{d-1}\}$.

We now consider Ty_1. Since $y_1 \in P$, $Ty_1 \in J^s$, that is,

$$Ty_1 = u_1 \cdots u_s U \text{ with } u_j \in \mathcal{G}(J).$$

First, note that y_1 does not divide U, since if it did we would then have $T = u_1 \cdots u_s (U/y_1) \in J^s$, a contradiction. Since $Ty_1 = T' y_1^{e_1+1} y_2^{e_2} \cdots y_{d-1}^{e_{d-1}} y_d^{e_d+c} x^{b+c}$, this means that (at least) $e_1 + 1$ of the generators $u_1, \ldots, u_s$ are divisible by y_1. We may assume that after reordering that these generators are $u_1, \ldots, u_{e_1+1}$.

We next observe that x also does not divide U. To see why, suppose that $U = xU'$. As noted above, $u_1 = y_1 y_{i_2} \cdots y_{i_{d-t+1}} m$ for some monomial m not divisible by x. Note that $(u_1 x)/y_1 = x y_{i_2} \cdots y_{i_{d-t+1}} m$ will also be a non-minimal generator of J. This means that

$$\begin{aligned} Ty_1 &= u_1 \cdots u_s U = (y_1 y_{i_2} \cdots y_{i_{d-t+1}} m) u_2 \cdots u_s (xU') \\ &= (x y_{i_2} \cdots y_{i_{d-t+1}} m) u_2 \cdots u_s (y_1 U'). \end{aligned}$$

If we now cancel y_1 from both sides, this implies that $T \in J^s$, a contradiction. So x cannot divide U, and thus at least $b + c$ of $u_1, \ldots, u_s$ are divisible by x. By Lemma 4.3, they cannot be among $u_1, \ldots, u_{e_1+1}$ since these are all divisible by y_1. Let us say that they are $u_{e_1+2}, \ldots, u_{e_1+b+c+1}$. To summarize, we now have

$$Ty_1 = \underbrace{u_1 \cdots u_{e_1+1}}_{\text{all divisible by } y_1} \cdot \underbrace{u_{e_1+2} \cdots u_{e_1+1+b+c}}_{\text{all divisible by } x} \cdots u_s U.$$

We finish the proof by counting the degrees of the variables $y_2, \ldots, y_d$ in Ty_1. There are two cases to consider: (*Case 1*) there is a generator among $u_1, \ldots, u_s$ of type (iii); and (*Case 2*) there is no generator among $u_1, \ldots, u_s$ of type (iii).

Case 1: Suppose there is some $u_j = x y_d m$. Then y_d must divide every generator among $u_1, \ldots, u_s$ of type (i). To see why, suppose that there is some generator $u_r = y_1 y_{i_2} \cdots y_{i_{d-t+1}} m'$ with $y_{i_\ell} \neq y_d$ for all $2 \leq \ell \leq d - t + 1$.

$$\begin{aligned} Ty_1 = u_1 \cdots u_r \cdots u_j \cdots u_s U &= u_1 \cdots (y_1 y_{i_2} \cdots y_{i_{d-t+1}} m') \cdots (x y_d m) \cdots u_s U \\ &= u_1 \cdots (x m') \cdots (y_{i_2} \cdots y_{i_{d-t+1}} y_d m) \cdots u_s (y_1 U). \end{aligned}$$

Note that $xm', y_{i_2} \cdots y_{i_{d-t+1}} y_d m \in \mathcal{G}(J)$. If we cancel y_1 from both sides, we get $T \in J^s$, which is a contradiction.

Similarly, suppose that there is some generator $u_r = y_{i_1} \ldots y_{i_{d-t+1}} m'$ with $y_{i_\ell} \neq y_1, y_d$ for all $1 \leq \ell \leq d$, and let $u_1 = y_1 y_{k_2} \cdots y_{k_{d-t}} y_d m''$ (since u_1 is divisible by y_1, it must also be divisible by y_d by above). Since $y_{i_\ell} \neq y_1$ for all $1 \leq \ell \leq d-t+1$, there is some variable among $y_{i_1}, \ldots, y_{i_{d-t+1}}$ which does not divide u_1. Without loss of generality, assume that y_{i_1} does not divide u_1. Then

$$\begin{aligned} Ty_1 &= u_1 \cdots u_r \cdots u_j \cdots u_s U \\ &= (y_1 y_{k_2} \cdots y_{k_{d-t}} y_d m'') \cdots (x y_d m) \cdots (y_{i_1} y_{i_2} \cdots y_{i_{d-t+1}} m') \cdots u_s U \\ &= (y_{i_1} y_{k_2} \cdots y_{k_{d-t}} y_d m'') \cdots (y_{i_2} \cdots y_{i_{d-t+1}} y_d m) \cdots (x m') \cdots u_s (y_1 U). \end{aligned}$$

The monomials xm', $y_{i_2} \cdots y_{i_{d-t+1}} y_d m$, and $y_{i_1} y_{k_2} \cdots y_{k_{d-t}} y_d m''$ are generators of J, so if we cancel y_1 from both sides this leads to the contradiction $T \in J^s$. So if there is some $u_j = x y_d m$, then every generator of type (i) among $u_1, \ldots, u_s$ is divisible by y_d.

Now consider the monomials $u_1, \ldots, u_{e_1+1}$. After relabeling, we may assume that $y_1, \ldots, y_j$ and y_d divide all of $u_1, \ldots, u_{e_1+1}$, and that each of the remaining variables $y_{j+1}, \ldots, y_{d-1}$ do not divide at least one of the generators $u_1, \ldots, u_{e_1+1}$. We now count the number of times that the variables $y_{j+1}, \ldots, y_{d-1}$ occur in the generators $u_1, \ldots, u_s$. Each of $u_1, \ldots u_{e_1+1}$ are divisible by exactly $d - t + 1$ of the variables $y_1, \ldots, y_d$ including

$y_1, \ldots, y_j$ and y_d. Therefore exactly $d-t+1-(j+1) = d-t-j$ of the variables $y_{j+1}, \ldots, y_{d-1}$ divide each of $u_1, \ldots, u_{e_1+1}$. In addition, the variables $y_{j+1}, \ldots, y_{d-1}$ may divide each of the monomials $u_{e_1+b+c+2}, \ldots, u_s$ (there are $s-(e_1+b+c+1) = a-e_1-1$ such monomials). Since y_d divides every generator of type (i) in the list $u_1, \ldots, u_s$, at most $d-t$ of the variables $y_{j+1}, \ldots, y_{d-1}$ divide each of the generators $u_{e_1+b+c+2}, \ldots, u_s$. In total, the number of times that the variables $y_{j+1}, \ldots, y_{d-1}$ divide the monomials $u_1, \ldots, u_s$ is at most

$$(d-t-j)(e_1+1) + (d-t)(a-e_1-1) = (d-t)a - je_1 - j.$$

On the other hand, since $T = T'y_1^{e_1} \cdots y_{d-1}^{e_{d-1}} y_d^{e_d+c} x^{b+c}$, the number of times that the variables $y_{j+1}, \ldots, y_{d-1}$ divide T is at least

$$\begin{array}{rcl} e_{j+1} + \cdots + e_{d-1} & = & e_1 + \cdots + e_d - (e_1 + \cdots + e_j + e_d) \\ & = & (d-t+1)a - (e_1 + \cdots + e_j + e_d). \end{array}$$

Since $e_1 = \max\{e_1, e_2, \ldots, e_{d-1}\}$ we have $e_1 + e_2 + \cdots + e_j \leq je_1$. So

$$\begin{array}{rcl} (d-t+1)a - (e_1 + \cdots + e_j + e_d) & \geq & (d-t+1)a - (je_1 + e_d) \\ & = & (d-t+1)a - je_1 - e_d. \end{array}$$

And since e_d is the number of times that the variable y_d appears among the (square-free) monomials $m_1, \ldots, m_a$ we have $a \geq e_d$. So

$$(d-t+1)a - je_1 - e_d \quad \geq \quad (d-t+1)a - je_1 - a = (d-t)a - je_1.$$

Since $j \geq 1$, this number is larger than the number of times that the variables $y_{j+1}, \ldots, y_{d-1}$ divide $u_1, \ldots, u_s$. Therefore, there must be some y_k with $j+1 \leq k \leq d-1$ which divides U. Let $U = y_k U'$. By assumption, there is some monomial among $u_1, \ldots, u_{e_1+1}$ which is not divisible by y_k. Without loss of generality, say $y_k \nmid u_1$. Then $u_1 = y_1 y_{i_2} \cdots y_{i_{d-t+1}} m'$ for some monomial m' with $y_{i_\ell} \neq y_k$ for all $2 \leq \ell \leq d-t+1$. Then

$$\begin{array}{rcl} Ty_1 = u_1 \ldots u_s U & = & (y_1 y_{i_2} \cdots y_{i_{d-t+1}} m') u_2 \cdots u_s (y_k U') \\ & = & (y_k y_{i_2} \cdots y_{i_{d-t+1}} m') u_2 \cdots u_s (y_1 U'). \end{array}$$

Since $y_k y_{i_2} \cdots y_{i_{d-t+1}} m' \in \mathcal{G}(J)$ this implies that $T \in J^s$, which is a contradiction. So $w \notin P$, and thus $\Gamma_P = K_{1,d}$, as desired.

Case 2: Suppose that no generator among $u_1, \ldots, u_s$ is of the form $xy_d m'$ (which implies $c = 0$). Assume again that each of the variables $y_1, \ldots, y_j$ with $1 \leq j < d$ divides each of the monomials $u_1, \ldots, u_{e_1+1}$ and that the variables $y_{j+1}, \ldots, y_{d-1}$ do not. Note that y_d may or may not divide every monomial in $u_1, \ldots, u_{e+1}$. We will count the variables $y_{j+1}, \ldots, y_d$. We saw in the previous case that we arrive at a contradiction if we assume that the variable y_d divides every minimal generator of type (i) in the list $u_1, \ldots, u_s$. Therefore we may assume that there is some monomial of type (i) among $u_1, \ldots, u_{e_1+1}, u_{e_1+b+2}, \ldots, u_s$ which is of type (i) and which is not divisible by y_d.

Now $(d-t+1-j)$ of the variables $y_{j+1}, \ldots, y_d$ divide each of the monomials $u_1, \ldots, u_{e_1+1}$. In addition, at most $(d-t+1)$ of the variables $y_{j+1}, \ldots, y_d$ divide each of the monomials

$u_{e_1+b+2},\dots,u_s$. In total the number of times that the variables $y_{j+1},\dots,y_d$ divide the monomials $u_1,\dots,u_s$ is at most

$$(d-t+1-j)(e_1+1)+(d-t+1)(s-(b+e_1+1))=(d-t+1)a-je_1-j.$$

On the other hand, since $T=T'y_1^{e_1}\cdots y_{d-1}^{e_{d-1}}y_d^{e_d}x^b$ (because $c=0$ in this case), the number of times the variables $y_{j+1},\dots,y_d$ divide Ty_1 is at least

$$\begin{aligned} e_{j+1}+\cdots+e_d &= e_1+\cdots+e_d-(e_1+\cdots+e_j) \\ &= (d-t+1)a-(e_1+\cdots+e_j) \\ &\geq (d-t+1)a-je_1 \end{aligned}$$

because $e_1=\max\{e_1,\dots,e_{d-1}\}$.

Since $j\geq 1$, this number is strictly greater than the number of times that $y_{j+1},\dots,y_d$ divide the monomials $u_1,\dots,u_s$. Therefore there is some y_k with $j+1\leq k\leq d$ which divides U. If $k\neq d$, then we know that there is some monomial among $u_1,\dots,u_{e_1+1}$ which is not divisible by y_k. Without loss of generality we may assume that y_k does not divide $u_1=y_1y_{i_2}\cdots y_{i_{d-t+1}}m'$. Then

$$\begin{aligned} Ty_1=u_1\cdots u_sU &= (y_1y_{i_2}\cdots y_{i_{d-t+1}}m')u_2\cdots u_s(y_kU') \\ &= (y_ky_{i_2}\cdots y_{i_{d-t+1}}m')u_2\cdots u_s(y_1U'). \end{aligned}$$

Since $y_ky_{i_2}\cdots y_{i_{d-t+1}}m'\in\mathcal{G}(J)$, this implies that $T\in J^s$ which is a contradiction.

Finally, assume that none of $y_{j+1},\dots,y_{d-1}$ divide U. Then y_d must divide U. Let $U=y_dU'$. If there is some monomial among $u_1,\dots,u_{e_1+1}$ which is not divisible by y_d then we arrive at a contradiction as above. If y_d divides each of $u_1,\dots,u_{e_1+1}$, then there is some monomial in the list $u_{e_1+b+2},\dots,u_s$ which is not divisible by y_d. Without loss of generality, assume u_s is not divisible by y_d. So $u_s=y_{k_1}\cdots y_{k_{d-t+1}}m$, where $y_{k_\ell}\neq y_d$ for all $1\leq\ell\leq d-t+1$ and $u_1=y_1y_{i_2}\cdots y_{i_{d-t}}y_dm'$. Since y_d divides u_1 and does not divide u_s there is at least one of the variables $y_{k_1},\dots,y_{k_{d-t+1}}$ which does not divide u_1. Assume that y_{k_1} does not divide u_1. Then

$$\begin{aligned} Ty_1 &= u_1\cdots u_sU \\ &= (y_1y_{i_2}\cdots y_{i_{d-t}}y_dm')u_2\cdots u_{s-1}(y_{k_1}y_{k_2}\cdots y_{k_{d-t+1}}m)(y_dU') \\ &= (y_{k_1}y_{i_2}\cdots y_{i_{d-t}}y_dm')u_2\cdots u_{s-1}(y_{k_2}\cdots y_{k_{d-t+1}}y_dm)(y_1U'). \end{aligned}$$

Since $y_{k_1}y_{i_2}\cdots y_{i_{d-t}}y_dm'$ and $y_{k_2}\cdots y_{k_{d-t+1}}y_dm$ are also minimal generators of J, this implies that T is an element of J^s which is a contradiction.

Therefore the associated prime P cannot be of the form $P=\langle y_1,\dots,y_d,x,w,\dots\rangle$. In other words, $\deg(y_d)=1$, so $\Gamma_P=K_{1,d}$ is a star graph as desired. □

We can now prove Theorem 1.2.

Proof. (of Theorem 1.2) The persistence property is immediate from our description of the sets $\mathrm{Ass}(J_t(\Gamma)^s)$ in Theorem 4.1. When $t=1$, $\mathrm{astab}(J_1(\Gamma))=1$ since Γ is bipartite. So the result follows from [7]. When $t\geq 2$, let x be a vertex with $\deg x=\Delta(\Gamma)$,

i.e., a vertex of maximal degree. Let $P = \{x\} \cup N(x)$. Then $\Gamma_P = K_{1,\Delta(\Gamma)}$. If we abuse notation, and let P also denote the ideal generated by the variables corresponding to the vertices in P, then $P \in \operatorname{Ass}(J_t(\Gamma)^s)$ if and only if $s(t-1) \geq \Delta(\Gamma) - 1$. So $\operatorname{astab}(J_t(\Gamma)) \geq \min\{s \mid s(t-1) \geq \Delta(\Gamma) - 1\}$.

Let $s_0 = \min\{s \mid s(t-1) \geq \Delta(\Gamma)-1\}$ and suppose that $\operatorname{astab}(J_t(\Gamma)) > s_0$. Because $J_t(\Gamma)$ has the persistence property, that means that there is a $P \in \operatorname{Ass}(J_t(\Gamma)^s)\backslash\operatorname{Ass}(J_t(\Gamma)^{s_0})$ with $s > s_0$. We can assume s is the smallest such integer with this property. By Theorem 4.1, $\Gamma_P = K_{1,r}$, and by Theorem 3.1, we must have $s(t-1) \geq r-1$. Since $P \notin \operatorname{Ass}(J_t(\Gamma)^{s_0})$, we must have $s_0(t-1) \not\geq r-1$. But this means that $r > \Delta(\Gamma)$, which implies that Γ has a vertex of degree greater than $\Delta(\Gamma)$, a contradiction. □

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